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XLIII. *A Letter to James West, Esq; President of the Royal Society, containing the Investigations of Twenty Cases of Compound Interest, by J. Robertson, Lib. R. S.*

S I R,

Read Dec. 13, 1770. **I**T is well known that the consideration of Compound Interest is of great utility in the Computations respecting the values of Pensions, Annuities, Reversions, and other affairs relating to Money concerns; and therefore, many Mathematicians have bestowed some time on this useful subject, and have endeavoured to discover practical methods for solving the various cases that might occur: And in these inquiries, the use of Logarithms has been found of singular service in facilitating the operations.

The late William Jones, Esq; F. R. S. among the variety of Mathematical matters to which he gave attention, considered the business of Compound Interest fully, and did, many years ago, cause to be engraved on a Copper-plate, more cases in Interest than had been exhibited before that time: Several copies of impressions from that plate were distributed among his friends; to whom it appeared that he had treated this subject in a more extensive manner than had been done by other Mathematicians.

The

The Theorems, or Rules, for the Cases of Compound Interest, without their investigations, were inserted by Mr. Jones, in the quarto edition of Logarithms, published by Gardiner; and the Rules were also communicated to Mr. Dodson, who published them, by Mr. Jones's leave, with examples to illustrate the use of his Antilogarithmic Tables: But the investigations of these Theorems not having yet been made public, it is apprehended that Gentlemen curious in these speculations would be pleased to see them: And should the following Essay be approved of by you, who have been for many years acquainted with the Finances of this Nation, and the importance of a ready knowledge in computing the Interest on the National Loans, you will be pleased to communicate it to the Royal Society.

I am, S I R,

Your most humble servant,

John Robertson.

In this subject five particulars are taken into consideration.

- 1st, The Annuity, Rent, or Pension.
- 2d, The Times that Annuity, Rent, or Pension is to continue.
- 3d, The Rate of Interest used in the computation.
- 4th, The Amount of those Rents, and their Interest, when they are forborn to be received any times after they are due.
- 5th, The present worth of those Rents, some times before they are due; or, of a Sum to be received before it is due, Discount being allowed.

And

And the investigations naturally fall under two heads.

First, The Consideration of Amounts.

Secondly, The Consideration of Discounts.

Under the first head an Equation is to be obtained between the Annuity, Time, Rate and Amount, from the known proportion that subsists between sums of money put to interest, during the same length of time, and the amounts of the principal and interest together.

Under the second head another Equation is to be formed between the Annuity, Rate, Time and present Worth, from the known proportion that subsists between the sums discounted, and their present worths, when done for the same time.

As these Equations involve quantities common to both of them, therefore other Equations may be thence deduced, containing all the five terms before specified. And hence, any three of the five terms being given, the other two are to be found, which admits of 20 Cases.

Some of these Cases will produce Affected Equations, where the index of the highest power of the unknown quantity will be the number of times the Rent is to continue, or to be paid: Therefore, the solution of those Cases will be given by a method of Approximation, as no better way has yet been discovered for the solution of Affected Equations, in numbers, above the third or fourth degree.

In the following investigations,

Let a = Annuity, Rent, or Pension.

n = Number of times that interest is to be paid
for the annuity, or sum lent.

r = Rate of interest of 1 £ . for 1 time.

m = Amount of the annuity, or sum lent for
 n -times, at r interest.

p = Principal sum used, or present worth of a
sum before it is due.

Of Compound Interest.

First, In Amounts,

Let $q = 1 + r$ = Amount of 1 £ . for 1 time.

Now, a = last year's amount.

And $1 : q :: a : aq$ = last but one year's amount.

$1 : q :: aq : aq^2$ = last but two year's amount.

$1 : q :: aq^2 : aq^3$ = last but 3.

And so on to aq^{n-1} = first year's amount.

Therefore $a + aq + aq^2 + aq^3$ &c. $+ aq^{n-1} = m$.

But $a : aq :: m - aq^{n-1} : m - a$. Euc. 12. v.

Then $m - a = mq - aq^n$

Or $m - a = m + mr - aq^n$

Therefore $aq^n - a = mr$

Put $A = q^n$.

Then $A - 1 \times a = mr$

Therefore $A = \frac{mr + a}{a}$

Hence $a = \frac{mr}{A - 1}$

$m = \frac{A - 1 \times a}{r}$

$r = \frac{A - 1 \times a}{m}$

Second,

Second, In Discounts.

Since $q : 1 :: a : \frac{a}{q} = 1^{\text{st}} \text{ year's present worth.}$

$q : 1 :: \frac{a}{q} : \frac{a}{q^2} = 2^{\text{d}} \text{ year's present worth.}$

$q : 1 :: \frac{a}{q^2} : \frac{a}{q^3} = 3^{\text{d}}.$

And so on to $\frac{a}{q^n} = n^{\text{th}} \text{ years present worth.}$

Therefore $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3} +, \&c. \text{ to } \frac{a}{q^n} = p.$

But $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}. \quad \text{Euc. 12. v.}$

Therefore $\frac{pq^n - a}{q^n} = pq - a.$

Or $\frac{pq^n - a}{q^n} = p + pr - a.$

Therefore $aq^n - a = prq^n.$

Or $\overline{A - 1} \times a = prA.$

Therefore $A = \frac{a}{a - rp}$

Hence $\overline{A - 1} \times a = mr = prA.$

Therefore $A = \frac{mr + a}{a} = \frac{a}{a - rp} = \frac{m}{p} = \overline{1 + r}^n.$

Case I. Given a, m, p : Required $r.$

Since $A = \frac{m}{p}.$

Therefore $r = \frac{\overline{A - 1} \times a}{m}.$

Case II.

Cafe II. Given p, m, n : Required r .

Since $\overline{r+1}^n = \frac{m}{p}$.

Therefore $r = \left(\frac{m}{p} \right)^{\frac{1}{n}} - 1$.

Cafe III. Given a, m, n : Required r .

Since $\overline{A-1} \times a = mr$.

Therefore $\left(\frac{A-1}{nr} \right) \frac{m}{na} = \frac{\overline{1+r}^n - 1}{nr}$.

Now $\overline{1+r}^n = 1 + nr + n \times \frac{n-1}{2} r^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} r^3$,
&c.

Therefore $\frac{m}{na} = 1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2$, &c.

And $\left(\frac{m}{na} \right)^{\frac{2}{n-1}} = \overline{1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2}^{\frac{2}{n-1}}$;

which, by the Binomial Theorem, will become
 $= 1 + r + \frac{n+1}{12} r^2$ (nearly).

Let $D = \left(\frac{m}{na} \right)^{\frac{2}{n-1}} = 1 + r + \frac{n+1}{12} r^2$,

Then $r^2 + \frac{12}{n+1} r = \overline{D-1} \times \frac{12}{n+1}$, Let $2E = \frac{12}{n+1}$,

Then $r + E = \left(\sqrt{2 \times D - 1 + E \times E} \right) F$.
Therefore $r = F - E$.

In this Solution, 1st, find $D = \left(\frac{m}{na} \right)^{\frac{2}{n-1}}$,

2d, find $E = \frac{6}{n+1}$,

3d, find $F = \sqrt{2 \times D - 1 + E \times E}$,

4th, find $r = F - E$.

Cafe IV. Given a, p, n ; Required r .

Since $\overline{A - 1} \times a = Apr$.

Therefore $\frac{p}{na} = \frac{A-1}{Anr} = \frac{1-A^{-1}}{nr} = \frac{1-\overline{1+r}^{-n}}{nr}$.

Now, $\overline{1+r}^{-n} = 1 - nr + n \times \frac{n+1}{2} r^2 - n \times \frac{n+1}{2} \times \frac{n+2}{3} r^3 +$
 $\&c,$

Therefore $\frac{p}{na} = 1 - \frac{n+1}{2} r + \frac{n+1}{2} \times \frac{n+2}{3} r^2$, nearly.

Then $\left[\frac{p}{na} \right]^{-\frac{2}{n+1}} = \left[1 - \frac{n+1}{2} r + \frac{n+1}{2} \times \frac{n+2}{3} r^2 \right]^{-\frac{2}{n+1}}$;

which, by the Binomial Theorem, will become
 $= 1 + r - \frac{n-1}{12} rr$, nearly.

Now, $\left[\frac{p}{na} \right]^{-\frac{2}{n+1}} = \left[\frac{na}{p} \right]^{\frac{2}{n+1}}$.

Let $G = \left(\left[\frac{na}{p} \right]^{\frac{2}{n+1}} = \right) 1 + r - \frac{n-1}{12} r^2$.

Therefore $rr - \frac{12}{n-1} r = -\overline{G-1} \times \frac{12}{n-1}$.

Let $2H = \frac{12}{n-1}$.

Then $r - H = \left(\sqrt{H-2 \cdot \overline{G-1} \times H} = \right) K$.

Therefore $r = H - K$.

Cafe V. Given a, n, r ; Required m .

Since $\overline{A - 1} \times a = mr$. Therefore $m = \frac{\overline{A - 1} \times a}{r}$.

Cafe VI.

Cafe VI. Given p, n, r ; Required m .

Since $A = \frac{m}{p}$. Therefore $m = pA$.

Cafe VII. Given a, p, r ; Required m .

Since $\frac{m}{p} = \frac{a}{a - rp}$. Therefore $m = \frac{ap}{a - rp}$.

Cafe VIII. Given a, p, n ; Required m .

Find $G = \frac{\overline{na}^{\frac{2}{n+1}}}{p}$; $H = \frac{3}{\frac{1}{2}n - 1}$; $K = \sqrt{H - 2G - 1} \times H$.

Now, $H - K = r$ by 4th. But $\overline{r+1}^n = \frac{m}{p}$.

Therefore $m = \overline{r+1}^n \times p$.

Cafe IX. Given a, n, r ; Required p .

Since $\overline{A-1} \times a = prA$. Therefore $p = \frac{\overline{A-1} \times a}{Ar}$.

Cafe X. Given m, n, r ; Required p .

Since $A = \frac{m}{p}$. Therefore $p = \frac{m}{A}$.

Cafe XI. Given a, m, r ; Required p .

Since $\frac{mr+a}{a} = \frac{m}{p}$. Therefore $p = \frac{ma}{mr+a}$.

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Cafe XII.

Cafe XII. Given a, m, n ; Required p .

$$\text{Find } D = \frac{\overline{m}^{\frac{2}{n-1}}}{na}; E = \frac{3}{\frac{1}{2}n+1}; F = \sqrt{2D-1+E \times E}$$

Now, $F - E = r$, by 3d. But $\overline{1+r}^n = \frac{m}{p}$.

$$\text{Therefore } p = \frac{m}{1+r^n}.$$

Cafe XIII. Given p, n, r ; Required a .

$$\text{Since } \overline{A-1} \times a = prA. \text{ Therefore } a = \frac{prA}{A-1}.$$

Cafe XIV. Given p, m, n ; Required a .

$$\text{Since } A = \overline{r+1}^n = \frac{m}{p}. \text{ Therefore } r+1 = \overline{\frac{m}{p}}^{\frac{1}{n}}.$$

Hence $A-1$, and $B-1 (=r)$ are known.

$$\text{Therefore } a = \frac{m \times \overline{B-1}}{A-1}.$$

Cafe XV. Given m, n, r ; Required a .

$$\text{Since } \overline{A-1} \times a = mr. \text{ Therefore } a = \frac{mr}{A-1}.$$

Cafe XVI. Given p, m, r ; Required a .

$$\text{Since } A = \frac{m}{p}, A-1 \text{ is given.}$$

$$\text{Therefore } a = \frac{mr}{A-1}.$$

Cafe XVII.

Cafe XVII. Given m, p, r ; Required n .

Put L , for logarithm; and \bar{L} , for arith. comp. of a log.

$$\text{Since } \overline{1+r}^n = \frac{m}{p}. \quad \text{Therefore } n = \frac{L, m + \bar{L}, p}{L, r + 1}.$$

Cafe XVIII. Given a, p, r ; Required n .

$$\text{Since } \overline{1+r}^n = \frac{a}{a-pr}. \quad \text{Therefore } n = \frac{L, a - L, \overline{a-pr}}{L, r + 1}.$$

Cafe XIX. Given a, p, m ; Required n .

$$\text{Since } A = (\overline{r+1})^n = \frac{m}{p},$$

$$\text{Then } (L, \overline{r+1})^n = L, m - L, p = L, A;$$

Hence A , and $A - 1$, are known.

$$\text{Also } L, \overline{A-1} + L, a + \bar{L}, m = L, \overline{B-1}$$

(by Cafe 14th):

Hence $r = (B - 1)$, and $r + 1$, are known.

$$\text{Therefore } n = \frac{L, A}{L, r + 1}.$$

Cafe XX. Given a, m, r ; Required n .

$$\text{Since } (A =) \frac{mr+a}{a} = \overline{1+r}^n.$$

$$\text{Therefore } n = \frac{L, \overline{mr+a} - L, a}{L, 1+r}.$$